Breaking Classical Rules

ONE-THIRD ANGLE IN TRIGONOMETRY

New method for Cubic Equation

BHAVA NATH DAHAL

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One-third Angle

in

Trigonometry

New method for Cubic or Higher-degree Equation

Bhava Nath Dahal

[Please read 'Request from the author']

This is abstract of Book 4 of Breaking Classical Rules in Trigonometry series:

Exact Values in Trigonometry: Five New Techniques (Vol I)

Precise-Rewritten Method: Exact Values in Trigonometry (Vol II)

Trigonometric Table for all angles: and polygon (Vol III)

One-third Angle in Trigonometry: New method for Cubic Equation (Vol IV)

Request from the author

This article or books in this series may be beneficial to the scholars. However, due to my low skill in mathematics and modern technologies, I need remarkable support to professional supervision, quality control, formatting, copy-editing etc. I am afraid, the concept I could contribute may be lost without good support for professionals or professional institution. I request someone mathematicians or mathematics institution or universities to support and supervise these works.

Introduction

Mathematicians and scholars have handled the complexity of computing trigonometric values of one-third of a known angle. The contributory for this complexity is the involvement of difficult cubic equation in the trigonometric formula. There are many methods to resolve the complex cubic equations. Guru Gerolamo Cardano's (1501-1576) method is one of best of them. There is somewhat similar method developed by Orson Pratt, Sen. (1811 –1881). In the addition of the complexity of original cubic equation, the methods comparatively seems complex procedure themselves. We are trying to resolve this complexity of solving cubic equation involve in the one-third of trigonometric values.

It is better to remind here, a person having knowledge of either 'Precise-Rewritten method for exact values in trigonometric ratios' or 'Arc – Line method' can compute any angle, irrespective of any ratio to another angle.

General pattern of cubic equation

An equation having cube (3) in most of exponential power is cubic equation. General from of cubic equation is:

$ax^3 + bx^2 + cx + d = 0$ Eq. 1

In this equation, the biggest power on x is cube (x^3). There are, normally, three roots in variable (x) within a cubic equation. Those three roots may be same or different each other. Sometimes, its roots may be undefined too. Solving cubic equation may be tough for many cases. Methods available for solving cubic equation passes through imaginary roots too.

Solving cubic equation having integers as roots.

As stated above, general cubic equation is in form of $ax^3 + bx^2 + cx + d = 0$. We need to solve this equation to value of x. Of course, x will have three roots (as denoted by p, q and r above).

Following will be the general pattern for solving cubic equation:

First of all, make x^3 having constant of 1 as:

$x^3 + b/a\ x^2 + c/a\ x + d/a = 0$ Eq. 2

Assuming our factors (roots) above as

$(x + p)\ (x + q)\ (x + r) = 0.$ Eq. 3

Expanding left hand side factors of this new equation:

$(x + p)\ (x + q)\ (x + r)$

$= [x^2 + (p+q)\ x + pq]\ (x + r)$

$= x^3 + (p+q+r)\ x^2 + (pq+qr+rp)\ x + pqr$

$= x^3 + (p+q+r)\ x^2 + [pq + r\ (p+q)]\ x + pqr$ Eq. 4

Comparing the above two equations, we come to end as:

$b/a = p+q+r$

$c/a = pq + r\ (p+q)$

$d/a = pqr$

This is the critical point of new method.

Prepare an equation from trigonometric formulae

We have, Sin 3A = 3 Sin A – 4 Sin^3 A. We knew, Sine of an angle will be half of double angle chord.

Re-arranging the equation taking chord for 2A as 'c/2' and A as 'a/2'.

$c/2 = 3\ a/2 - 4\ a^3/8$

$c = 3a - a^3$

Rearranging,

$a^3 - 3a + c = 0$ Eq. 5

This is cubic equation for Sin 3A. For higher degree, we need to modify the expression in similar format of a and c.

Concept of supplementary chord

A line from 0° to the given angle A° is called as chord (notation is 'a'). If a line drawn from point of A to 180°. This is supplementary chord (notation is 'b'). knowing since 200 years, chord and supplementary chord makes a right angle in A.

Using Guru Pythagoras famous formula of right–angled triangle for a unit circle, the relation between a chord (a) and supplementary chord (b) is:

$a = \sqrt{(4 - b^2)}$ or

$b = \sqrt{(4 - a^2)}$ Eq. 6

Three roots of cubic equation

Let us assume new notation for each of three roots of $a^3 - 3a + c = 0$ as p, q and r. In Sin 3A = 3 Sin A – 4 Sin^3 A, half of:

- One root will be Sin A (say root p/2). This will be the smallest root in quantity amongst three roots. This shall be *near to* one-third of c. Its mathematical value will be positive (+p).

  $p = c/3$ roughly Eq. 7

  after this first guess, corrected value of p will be conversion of original formula $c = 3a - a^3$ as:

  $3a = c - a^3$

  $a = (c - a^3)/3$

  therefore,

  $p = (c - p^3)/3$ [here, p is new p and p^3 is cube of earlier p]

- Second root (say root q/2) will be Sin (60° – A). This will be the second largest root in quantity amongst three roots. Its mathematical value will be positive (+q).

  $$q = \frac{\pm\sqrt{3}\times\sqrt{4-p^2}}{2} - \frac{p}{2}$$ (see below for proof)

- Third root (say root r/2) will be Sin (60° + A). Its mathematical value will be negative. In quantity, r will be **exactly** sum of p and q [with negative sign; i.e. $r = -(p + q)$]. This will be the largest root in quantity amongst three roots.

  $\sin(60^\circ + A) = -(p + q)$ Eq. 8

In summary:

Smallest root is p, $\sin A = +p/2$

Second root is q, $\sin(60^\circ - A) = +q/2$

Largest root is r, $\sin(60^\circ + A) = -(p + q)/2 = r/2$

This means roots of one-third formula are not one-third only but they are 60° ± one-third too[1].

New clue of Sin (60° ± A)

We found two roots of one-third angle (A) from triple angle (3A) is Sin (60° ± A). This opens new arena for trigonometric function.

For its proof, take back $a^3 - 3a - c = 0$ with three roots p, q and r.

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[1] This property is equally applicable to higher degree equation too. For example, one-fifth angle will be A/5, 36±A/5 and 72±A/5 with root dependency upon Sin 36° and Sin 72°. Similar expansion will be in 6th degree, 7th degree or nth degree equation.

$p + q + r = 0$ … … [1]

$pq + (p+q)\ r = -3$ … … [2]

$pqr = +c$ … … [3]

From [1],

$r = -(p+q)$ … … [4]

From [2],

$pq + (p+q)\ r = -3$

or, $pq - (p+q)^2 = -3$

or, $pq - (p^2 + 2pq + q^2) = -3$

or, $p^2 + pq + q^2 = 3$

or, $q^2 + qp + (p^2 - 3) = 0$ [assuming p is known.]

Therefore,

$$q = \frac{-p \pm \sqrt{p^2 - 4(p^2 - 3)}}{2}$$

$$q = \frac{-p \pm \sqrt{p^2 - 4p^2 + 12)}}{2}$$

$$q = \frac{-p \pm \sqrt{12 - 3p^2}}{2}$$

$$q = \frac{-p \pm \sqrt{3}\sqrt{4 - p^2}}{2} \qquad \dots \dots [5]$$

Fortunately, if we take, +, the value will be root q and if we take -, the value will be r.

Further analyzing [5] as;

$$q = \frac{\pm\sqrt{3}\sqrt{4 - p^2} - p}{2} \qquad \dots \dots [6]$$

Alternatively, we may convert it into Sine function as:

$$q = \frac{\pm\sqrt{3} \times \sqrt{4 - p^2}}{2} - \frac{p}{2} \quad \dots \dots [7]$$

$$q = +Sin\ 60° \times \sqrt{4 - p^2} - Cos\ 60° \times p \qquad \dots \dots [8]$$

$$r = -Sin\ 60° \times \sqrt{4 - p^2} - Cos\ 60° \times p \qquad \dots \dots [9]$$

Tips:

Formula for value of q resolve new method for roots in higher degree equation. In cubic equation is came as both Sin 60˚ and Cos 60˚ (√3/2, 1/2) basis. In higher-degree equations, it will appear with similar pattern. For example, in A/5, Sin 36˚ and Cos 36˚ (180˚/5) for q and r and Sin 72˚ and Cos 72˚ (360˚/5); and so on.

Where are these roots?

Let us assume a unit circle (having radius 1). We have known angle 3A˚ with known chord length (c). If we divide angle 3A into equal 3 parts, each angle will be A˚. Let us assume chord length of A˚ is a.

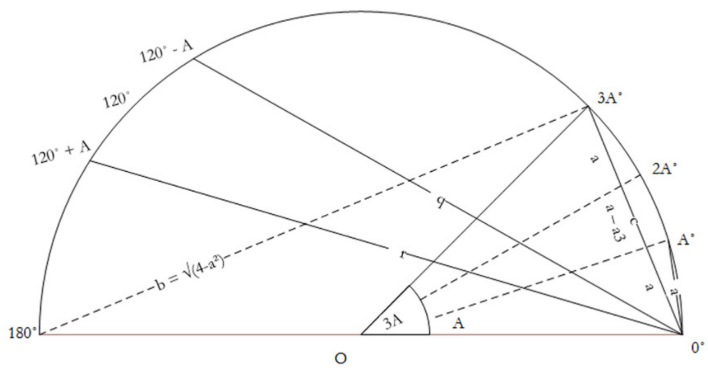

Figure 1 Roots of One-third angles

Dividing angle 3A into three equal parts, its chord will not have three equal length. In fact, chord 3A (noted as c) will have a, $a – a^3$ and a length.

Therefore, $a + a – a^3 + a = c$

And hence, $a^3 - 3a + c = 0$

Solving, above equation, we shall obtain three roots for a. we have denoted those three roots as p, q and r. In the picture above, chord representing (120˚ ± A) has shown as q and r. Chord representing A has shown as ‘a’ in place of ‘p’.

Finding Trigonometric values of one-third angle

Trigonometric cubic equation in form of $a^3 - 3a + c = 0$, has not all inputs as in general cubic equation in form of $ax^3 + bx^2 + cx + d = 0$. Therefore, we cannot use general method to solve these equations. For this, however, we can above process.

Example

Let us take an example of solving one-third of Sin 30˚

We know the value of Sin **30˚** is ½. Therefore, double of Sin **30˚** will be 2. This is the chord for **60˚.** Its equation will be:

$$a^3 - 3a + 1 = 0 \qquad \ldots \ldots \ldots [1]$$

Converting above equation into re-arranged format as given in:

$x^3 + b/a\, x^2 + c/a\, x + d/a = 0$

$x^3 + (p+q+r)\, x^2 + [pq + r\,(p+q)]\, x + pqr$

We shall obtain following new propositions:

$$b/a = p + q + r \quad = 0 \qquad \ldots \ldots [2]$$

$$c/a = pq + (p + q)\, r = -3 \qquad \ldots \ldots [3]$$

$$d/a = pqr \qquad = +1 \qquad \ldots \ldots [4]$$

Recalling above concept is good here.

$$r \qquad = -(p + q) \quad \ldots \ldots \ldots \ldots [5]$$

$$p < q < r \qquad \ldots \qquad \ldots \ldots \ldots [6]$$

We have taken limited accuracy of 14- digits after decimal point. For more accuracy, higher precision requires. For details in accuracy, see topic regarding accuracy.

Steps after above preparatory works:

Step 1: Value for p

Take p = one-third of known chord, this is rough root estimation.

$p = c/3$

$p = 1/3 = 0.333333333333333$

Step 2: Value for q

Using

$$q = \frac{+\sqrt{3}\times\sqrt{4-p^2}}{2} - \frac{p}{2}$$

In above p, we can compute value of q as (assuming √3 = 1.73205080756888):

q $= [1.73205080756888 \times \sqrt{(4–0.333333333333333^2)} – 0.333333333333333]/2$

$= 1.54115846099327$

Step 3: Value for r

Roughly, we computed value of two roots, p and q. in the third step, we need to compute root r. Root r shall be just sum of p and q with negative sign.

r = – (p + q)

= – (0.333333333333333 + 1.54115846099327)

= – 1.87449179432660

Alternatively, we can use following formula:

$$q = \frac{-\sqrt{3}\times\sqrt{4-p^2}}{2} - \frac{p}{2}$$

Step 4: Testing result of p, q and r

Though we know, this is our first calculation; its result is gestimated rough. It needs further calculations for correction. However, this testing step is critical to know whether we achieved the desired result or not.

There may be verities of techniques for testing result. In this short summary, we use testing with differences with earlier step. In this method, we compare new values of p, q or r with earlier value. If there is no change in values or if the changes in as much minor to be acceptable for the user-defined accuracy, we came to the end result of roots.

Step 5 Corrections to value for p

In none case, first route of calculation gives correct roots. We need to perform multiple repetition of calculations (sometimes called as iteration). In each process, step 1 to step 4 is same procedure. To start new process, we need to guess new value for p. New value for p shall be one-third of sum of Crd. 3A and cube of earlier p:

$p = (c+ p^3)/3$ Eq. 9

Where,

p is new p

Crd. 3A is known and given chord (in our case Crd. 60° i.e. 1)

p^3 is cube of rough p just in earlier step.

In our case, new value of p shall be:

$$p = (1+0.333333333333333^3)/3$$

$$= 0.345679012345679$$

From this new p = 0.345679012345679, repeat the process from step 1 to step 5 until testing result has not achieved.

Using above process until achieving the accurate value (see accuracy for detail), we obtained following table:

p	q	r
0.333333333333333	1.541158460993270	-1.874491794326600
0.345679012345679	1.533143939033410	-1.878822951379090
0.347102186947062	1.532215612105640	-1.879317799052700
0.347272948851898	1.532104163609850	-1.879377112461750
0.347293532356960	1.532090728803460	-1.879384261160420
0.347296014843951	1.532089108476930	-1.879385123320890
0.347296314265793	1.532088913043230	-1.879385227309030
0.347296350380446	1.532088889471070	-1.879385239851510
0.347296354736406	1.532088886627920	-1.879385241364320
0.347296355261799	1.532088886284990	-1.879385241546790
0.347296355325169	1.532088886243630	-1.879385241568800
0.347296355332812	1.532088886238640	-1.879385241571450
0.347296355333734	1.532088886238040	-1.879385241571770

0.347296355333845	1.532088886237970	-1.879385241571810
0.347296355333859	1.532088886237960	-1.879385241571820
0.347296355333860	1.532088886237960	-1.879385241571820
0.347296355333861	1.532088886237960	-1.879385241571820
0.347296355333861	1.532088886237960	-1.879385241571820

Above solution is accurate for 14-digits after decimal point.

Step 6: Finding chord length or value of Sine from roots

Form above table, we have obtained following three chords (removing negative sign of root r):

p = 0.347296355333861 [half of this is Sin A/3]

q = 1.532088886237960 [half of this is Sin (60˚ - A/3)]

r = 1.879385241571820 [half of this is Sin (60˚ + A/3)]

Our given angle (3A) was of Sin 30° = 1/2

Therefore,

Sin 30 = 1/2

= o.5

Sin 10° = 0.34729635533386/2

= 0.17364817766693

Sin 50 ° = 1.53208888623796/2

= 0.766044443118978

Sin 70 ° = 1.87938524157182/2

=0.93969262078591

Let us take another example as one-third angle of Sin 60°.

We know Sin 60° is is √3/2. From this information, we can directly prepare a cubic equation as:

$$a^3 - 3a + \sqrt{3} = 0$$

Now, we need to define the roots as p, q, and r. It will have the properties as p<q<r; r = - (p+q); and all the roots are in real roots in decimal number system.

Our first rough root, p = √3/3 = 1.73205080756888/3 = 0.577350269189626.

From p, we compute q from (√3 × (4 - $p^{2)}$ - p)/2. The value of q shall be 1.36963726058289.

From p and q, we compute value of r as

r = – (0.577350269189626 + 1.36963726058289)

= (1.9469875297725)

Our first round calculation ends here.

For the second round, we further estimate new value of p using earlier p = 0.577350269189626 and formula for new p as:

$$p = (1.73205080756888 + 0.577350269189626^3)/3$$

$$= 0.641500299099584$$

We have not tested our result, because we want to make a table for correction. Once the result within our desired-accuracy achieved, we stop further calculation. In that case, testing will be extra burden for users.

From this new p, we compute q as well as r and repeat the process until the value unchanged within our desired accuracy. The calculation table is as follows:

p	Q	r
0.577350269189626	1.369637260582890	-1.946987529772510
0.641500299099584	1.319785746031700	-1.961286045131280
0.665347566734150	1.300722819292770	-1.966070386026920
0.675530593903050	1.292493520698530	-1.968024114601580
0.680107836119171	1.288776950519770	-1.968884786638940
0.682210807185683	1.287065751697440	-1.969276558883120

0.683186540315370	1.286271009611650	-1.969457549927020
0.683641307630181	1.285900428519430	-1.969541736149610
0.683853708800456	1.285727309947930	-1.969581018748380
0.683953008611491	1.285646367095510	-1.969599375707000
0.683999453496931	1.285608506430490	-1.969607959927420
0.684021181505090	1.285590793935760	-1.969611975440850
0.684031347390755	1.285582506704220	-1.969613854094980
0.684036103926624	1.285578629156940	-1.969614733083570
0.684038329519926	1.285576814840220	-1.969615144360150
0.684039370890419	1.285575965908000	-1.969615336798420
0.684039858157200	1.285575568684650	-1.969615426841850
0.684040086154294	1.285575382819760	-1.969615468974050
0.684040192836570	1.285575295851580	-1.969615488688150
0.684040242754371	1.285575255158210	-1.969615497912580
0.684040266111460	1.285575236117340	-1.969615502228800
0.684040277040501	1.285575227207900	-1.969615504248410
0.684040282154320	1.285575223039080	-1.969615505193400
0.684040284547133	1.285575221088440	-1.969615505635570
0.684040285666757	1.285575220175720	-1.969615505842470
0.684040286190641	1.285575219748640	-1.969615505939280
0.684040286435773	1.285575219548810	-1.969615505984580

0.684040286550472	1.285575219455300	-1.969615506005780
0.684040286604141	1.285575219411550	-1.969615506015690
0.684040286629254	1.285575219391080	-1.969615506020340
0.684040286641004	1.285575219381500	-1.969615506022510
0.684040286646502	1.285575219377020	-1.969615506023520
0.684040286649075	1.285575219374920	-1.969615506024000
0.684040286650279	1.285575219373940	-1.969615506024220
0.684040286650842	1.285575219373480	-1.969615506024320
0.684040286651106	1.285575219373270	-1.969615506024370
0.684040286651229	1.285575219373170	-1.969615506024400
0.684040286651287	1.285575219373120	-1.969615506024410
0.684040286651314	1.285575219373100	-1.969615506024410
0.684040286651326	1.285575219373090	-1.969615506024410
0.684040286651332	1.285575219373080	-1.969615506024410
0.684040286651335	1.285575219373080	-1.969615506024420
0.684040286651336	1.285575219373080	-1.969615506024420
0.684040286651337	1.285575219373080	-1.969615506024420
0.684040286651337	1.285575219373080	-1.969615506024420

Therefore, three roots of one-third Sin 60˚ are:

p = 0.684040286651337

q = 1.28557521937308

r = 1.96961550602442

Therefore,

Sin 60° = 1.73205080756888/2

Sin 20° = 0.684040286651337/2

Sin 40° = 1.28557521937308/2

Sin 80° = 1.96961550602442/2

Accuracy of above procedure

Let us start answering this topic by question. What is the value of √2 or √3 or √5? Many scholars says √2 = 1.414 and √3 = 1.732. this answer is not wrong but not correct also. Because, these are candidate of irrational number (surds). They have not exact decimal precision not their pattern of number is in repeating nature.

Another example of 20/3 is 0.66666666…. Multiplying back with 3, the exact value is something other than 20. Similar the case with 1/3 and back multiplication by 3. In these cases, appropriate accuracy for 1 is 0. 99999$_9$ ... and vice versa.

Nevertheless, for practical purpose, we cannot speak or write infinitive digits; rather we set a fix-length; e.g. 5-digits, 12-digits, 15-digits, 25-digits etc. In the computer programming also, programmers set significant digits (SD). In that case, we assumed n numbers of 9's after decimal is 1. This is the user-defined *appropriate accuracy (Precise accuracy)* for our purpose.

In our calculations, we have used such surds in each step of computation. Therefore, users require fixing their precision level themselves.

Precise accuracy have two input parameters. The first is the number of digits handled during each process of calculation and the second is the number of calculations itself.

Firstly, for example, if we define the appropriate accuracy of 15-digits, we must use, at least, 15-digits after decimal in each step of calculation.

Secondly, we must perform such appropriate number of iteration, which the user-defined accuracy established. We can see above examples, number of calculations in different examples is not same. This is because of establishment of precise-accuracy at user-defined level. Above examples are based accuracy of 14-digits after decimals. To obtain Sin 20°/3 with 14-digits accurate, we need to perform far more than 100 iterations.

Finding Trigonometric values of A/n^{th} angle

Main objective of this work is to explore new idea of trigonometric values for one-third angle. Another objective of the book is to introduce new idea to solve cubic equation without using imaginary number. In the addition of it, same concept may be used for other angles (A/n^{th}) using similar concept. In each additional degree of equation, there will be one more root. All the roots are real root and represents some defined chord within the circle.

We noticed A/n^{th} angle is not n^{th} part only, but have $(360^\circ \pm A)/n^{th}$ part relation also. When we increase the value of n, we shall obtain additional roots vis-à-vis chords for new angle.

We have noticed, there is an easy method for cube root for trigonometric equation without imaginary number of complex procedure for depression. This would make scholars and professionals happy for their works.

We can, now, consume similar concept for higher degree equations.

One-fourth Angles (A/4)

Equation for one-fourth angle will have power 4 (so-called four-degree equation). Therefore, it will have four roots. Solving one-fourth angle through equation will be a fool task. Because, there are easy and simple classical method half-angles in half-series (A/2, A/4, A/8 etc.). though we use equation method, its five roots represent:

- $4A/4 = \pm A$
- $(360^\circ \pm 4A)/4 = 90^\circ \pm A$
- $(180^\circ \pm 4A)/4 = 45^\circ \pm A$

One-fifth Angles (A/5)

Equation for one-fifth angle will have power 5 (so-called five-degree equation). Therefore, it will have five roots. Those five roots will be as follows:

- $5A/5 = A$
- $(180^\circ \pm 5A)/5 = 36^\circ \pm A$
- $(360^\circ \pm 5A)/5 = 72^\circ \pm A$

As shown in earlier, A/n^{th} angle will expand down to many angles. We can obtain the trigonometric values accordingly. For the timing, expanding these angles is beyond the scope of this book.

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About Breaking Classical Rules: Mission 2050

Amateur effort is freedom of work. There will be no rains, limits, thresholds, budget constraints, or deadline pressures. In this line, this series of book breaking classical rules in trigonometry is a freedom works. The series breaks the classical system in the trigonometry. Therefore, this is Breaking Classical Rules.

In the breakings, there are five different methods for determination of trigonometric values:

> **Arc-Line method**- breaks complexity of determination of trigonometry. It is easy for learning within ½ hour and can be used by school scholars too.
>
> **Series-Method** – breaks series of all known angles into pieces and pieces of angles. The trigonometric values from Series-Method are exact values. It is easy for learning within one and a half hour and can be used by school scholars too.
>
> **Angle-Rewritten method** – breaks not only classical methods but also Series-Method too. It is comparatively complex for learning. It takes more than two hours for single sitting.
>
> **Precise-Rewritten method** – breaks all the existing system of determination of trigonometric values. It gives exact values for all angles in nested radical of 2. Even it gives exact values of all angles within user defined precision level, it is easy for learning. It takes less than two hours for single sitting.
>
> **Vasta-method** – breaks number system in the series. Trigonometric values from this method is least accurate than above methods.

Second breaking is that, we converted all the integer angles and polygons upto 120-gon (except 81-gon and 115-gon yet) into exact values and other arbitrary angles in exact radicals within precise accuracy level.

Solving cubic equation and higher degree equations as per this book also another **Breaking of Classical Rules**.

Changing the mind of people is a difficult task. If someone amateur challenges the classical system run since millennium, it is obvious it takes time to make its size. Consequently, adaptation of new methods may be targeted for 2050. Therefore, this is Mission 2050.

Hence the topic is named as '**Breaking Classical Rules: Mission 2050'.**

About the author

I am Bhava Nath Dahal, from Nepal, professional accountant and tax-expert. In mathematics, especially in trigonometry, I have published four books - Exact Values in Trigonometry: Five New Techniques, Precise-Rewritten method: Exact Values in trigonometry, Trigonometric Table for all angles: and polygon and this. Unpublished 'Breaking Classical Rules: Mission 2050' are working in pipeline.

One point require disclosing without any hesitation is that I am not a mathematician. Therefore, this is not a professional assignment, rather just an amateur interest. Nevertheless, this is new concept and possibly so simple too. Every step of study, writing, typing, formatting, uploading, proofreading and whatever required actions has done by single hand. So, any of the shortfall in this series of mine, with my apologies.

This article or books in this series may be beneficial to the scholars. However, due to my skill in mathematics, I need remarkable support to professional supervision, quality control, formatting, copy-editing etc. I am afraid, the concept I could contribute may be lost without good support for professionals or professional institution. I request someone mathematicians or mathematics institution or universities to support and supervise these works.

I request volunteer from different languages to translate this book with written permission. For direct access to me, I request professional commentaries through bhavanathdahal@gmail.com.

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Kojagrat Purnima

Kathmandu, Nepal